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# Growth and zeros of meromorphic solution of some linear difference equations ☆

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In this paper, we study growth and zeros of linear difference equations

$$P_n(z)f(z+n)+\cdots+P_1(z)f(z+1)+P_0(z)f(z)=F(z)$$

where $F(z), P_n(z), \dots, P_0(z)$ are polynomials with $FP_nP_0 \not\equiv 0$ and satisfy $\deg(P_n+\cdots+P_0)=\max\{\deg P_j\colon\ j=0,\dots,n\}\geqslant 1$. The corresponding homogeneous equation of the above equation is also investigated.



## 1. Introduction and results

In this paper, we use the basic notions of Nevanlinna's theory (see [10,17]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Recently, a number of papers (including [1–9,12–16]) focus on complex differences and difference equations. They obtain many new results on difference equations utilizing the value distribution theory of meromorphic functions.

Chiang and Feng [6] considered linear difference equations and obtained the following theorem.

**Theorem A.** *(See [6].) Let $P_0(z), \dots, P_n(z)$ be polynomials such that there exists an integer $l$, $0\leqslant l\leqslant n$ so that*

$$\deg(P_l) > \max_{0\leqslant j\leqslant n,\ j\neq l}\{\deg(P_j)\} \tag{1.1}$$

*holds. Suppose $f(z)$ is a meromorphic solution to*

$$P_n(z)y(z+n)+\cdots+P_1(z)y(z+1)+P_0(z)y(z)=0. \tag{1.2}$$

*Then we have $\sigma(f)\geqslant 1$.*

Ishizaki and Yanagihara [14] considered the growth of transcendental entire solutions of difference equation

$$Q_n(z)\Delta^n f(z)+\cdots+Q_1(z)\Delta f(z)+Q_0(z)f(z)=0, \tag{1.3}$$

where $Q_n, \dots, Q_0$ are polynomials, $\Delta f(z)=f(z+1)-f(z)$, $\Delta^n f(z)=\Delta(\Delta^{n-1}f(z))$, and obtain the following theorem.

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**Theorem B.** *(See [14].) Let $f(z)$ be transcendental entire solutions of* (1.3) *and of order $\chi < \frac{1}{2}$. Then we have*

$$\log M(r, f) = Lr^{\chi}\big(1 + o(1)\big),$$

*where a rational number $\chi$ is a slope of the Newton polygon for Eq.* (1.3)*, and $L > 0$ is a constant. In particular, we have $\chi > 0$.*

Comparing Theorem A with Theorem B, we see that although Eq. (1.2) can be rewritten as (1.3), the condition (1.1), the polynomial $P_l$ is dominating coefficient, guarantees that all transcendental meromorphic solutions of (1.2) satisfy $\sigma(f) \geqslant 1$.

The following Example 1 shows that under the general situation, if there is no dominating coefficient, $\sigma(f) \geqslant 1$ may not hold.

**Example 1.** (See [14].) The difference equation

$$\big(6z^2 + 19z + 15\big)\Delta^3 f(z) + (z+3)\Delta^2 f(z) - \Delta f(z) - f(z) = 0 \tag{1.4}$$

i.e.

$$\begin{aligned}&\big(6z^2 + 19z + 15\big) f(z+3) - \big(18z^2 + 56z + 42\big) f(z+2)\\&\quad + \big(18z^2 + 55z + 38\big) f(z+1) - \big(6z^2 + 18z + 12\big) f(z) = 0\end{aligned} \tag{1.5}$$

admits an entire solution of order $\frac{1}{3}$.

**Remark 1.1.** From Example 1, we see that in Eq. (1.5),

$$\deg P_3 = \cdots = \deg P_0 = 2,$$

there is no dominating coefficient. But we note that in Eq. (1.5)

$$\deg(P_3 + \cdots + P_0) = 0 < \max\{\deg P_j\colon\ j = 0, \ldots, 3\}.$$

Thus, it is natural to consider that whether the condition (1.1) can be replaced by the condition

$$\deg(P_n + \cdots + P_0) = \max\{\deg P_j\colon\ j = 0, \ldots, n\}?$$

The following Theorems 1.1 and 1.2 answer the question above.

**Theorem 1.1.** *Let $F(z), P_n(z), \ldots, P_0(z)$ be polynomials such that $F P_n P_0 \not\equiv 0$ and*

$$\deg(P_n + \cdots + P_0) = \max\{\deg P_j\colon\ j = 0, \ldots, n\} \geqslant 1. \tag{1.6}$$

*Then every finite order transcendental meromorphic solution $f(z)$ of equation*

$$P_n(z) f(z+n) + \cdots + P_1(z) f(z+1) + P_0(z) f(z) = F(z) \tag{1.7}$$

*satisfies $\sigma(f) \geqslant 1$ and $\lambda(f) = \sigma(f)$.*

**Theorem 1.2.** *Let $P_n(z), \ldots, P_0(z)$ be polynomials such that $P_n P_0 \not\equiv 0$ and satisfy* (1.6)*. Then every finite order meromorphic solution $f(z) (\not\equiv 0)$ of equation*

$$P_n(z) f(z+n) + \cdots + P_1(z) f(z+1) + P_0(z) f(z) = 0 \tag{1.8}$$

*satisfies $\sigma(f) \geqslant 1$, and $f(z)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often and $\lambda(f - a) = \sigma(f)$.*

**Remark 1.2.** Theorem 1.2 shows that the condition (1.6) guarantees that Eq. (1.8) has no rational solution.

**Remark 1.3.** Theorem A may be deduced from Theorem 1.2.

By Theorems 1.1 and 1.2, we clearly obtain the following corollary.

**Corollary 1.1.** *Let $F(z), P_n(z), \ldots, P_0(z)$ be polynomials such that $F P_n P_0 \not\equiv 0$ and satisfy* (1.6)*. Then* (1.7) *has at most one non-zero rational solution, all other meromorphic solutions of* (1.7) *satisfy $\sigma(f) \geqslant 1$.*

**Remark 1.4.** Theorem B shows that Eq. (1.3) (or (1.2)) may have a transcendental entire solution with order $< 1$. Thus, it is natural to ask if Eq. (1.3) (or (1.2)) has a meromorphic solution with infinitely many poles and order $< 1$?

The following Theorem 1.3 shows that Eq. (1.2) (or (1.3)) has no solution stated above.

**Theorem 1.3.** *Let $F(z), P_n(z), \dots, P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Suppose that $f(z)$ is a meromorphic solution with infinitely many poles of* (1.7) *(or* (1.8)*). Then $\sigma(f) \geqslant 1$.*

**Remark 1.5.** In Theorem 1.3, the condition (1.6) is omitted. Thus, we see that in Theorems 1.1 and 1.2, the condition (1.6) mainly guarantees that an entire solution of (1.7) (or (1.8)) satisfies $\sigma(f) \geqslant 1$.

**Example 2.** The equation

$$(z+3)f(z+2)+(z+2)^2 f(z+1)+\left(z^2-1\right)f(z)=2z^2+3z+4$$

has a rational solution $f(z)=\frac{z}{z+1}$. This shows that our Corollary 1.1 is sharp.

**Example 3.** The equation

$$\left(\frac{1}{2}z-1\right)f(z+2)-2(z-2)f(z)=0$$

has a solution $f(z)=2^z$, here $f(z)$ satisfies $\lambda(f-a)=\sigma(f)=1$ for any non-zero finite value $a$, and $f(z)$ has no zero. This shows that Theorem 1.2 is sharp.

**Example 4.** The equation

$$f(z+2)+z^2 f(z+1)-\left(z^2+1\right)f(z)=0$$

has a solution $f(z)=\tan(\pi z)$, here $\deg(P_0+P_1+P_2)=0<\{\deg P_j\colon\ j=0,1,2\}=2$ and $f(z)$ satisfies $\lambda(\frac{1}{f})=\sigma(f)=1$. The equation and its solution satisfy Theorem 1.3 (for Eq. (1.8)).

**Example 5.** The equation

$$f(z+2)+z^2 f(z+1)-\left(z^2+1\right)f(z)=z^2+2$$

has solutions $f_1(z)=z$ and $f_2(z)=\tan(\pi z)+z$, here $\deg(P_0+P_1+P_2)=0<\{\deg P_j\colon\ j=0,1,2\}=2$ and $f_2(z)$ satisfies $\lambda(\frac{1}{f_2})=\sigma(f_2)=1$. The equation and its solutions satisfy Theorem 1.3 (for Eq. (1.7)).

## 2. The proof of Theorem 1.1

We need following lemmas and remark to prove Theorem 1.1.

**Remark 2.1.** Following Hayman [11, pp. 75–76], we define an $\varepsilon$-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If $E$ is an $\varepsilon$-set then the set of $r \geqslant 1$ for which the circle $S(0,r)$ meets $E$ has finite logarithmic measure, and for almost all real $\theta$ the intersection of $E$ with the ray $\arg z=\theta$ is bounded.

**Lemma 2.2.** *(See [2].) Let $g$ be a function transcendental and meromorphic in the plane of order less than 1. Let $h>0$. Then there exists an $\varepsilon$-set $E$ such that*

$$\frac{g'(z+c)}{g(z+c)}\to 0, \qquad \frac{g(z+c)}{g(z)}\to 1 \quad \text{as } z\to\infty \text{ in } \mathbf{C}\setminus E,$$

*uniformly in $c$ for $|c|\leqslant h$. Further, $E$ may be chosen so that for large $z$ not in $E$ the function $g$ has no zeros or poles in $|\zeta-z|\leqslant h$.*

**Lemma 2.3.** *(See [7,16].) Let $w(z)$ be a non-constant finite order meromorphic solution of*

$$P(z,w)=0$$

*where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r,a)=S(r,w)$, then*

$$m\left(r,\frac{1}{w-a}\right)=S(r,w).$$

**Proof of Theorem 1.1.** Suppose that $f(z)$ is a finite order transcendental meromorphic solution of (1.7). We divide this proof into the following three cases.

**Case 1.** Suppose that $f(z)$ is a transcendental entire function. Now we suppose that $\sigma(f)<1$. By Lemma 2.2, there exists an $\varepsilon$-set $E$ such that

$$f(z+j)=f(z)\big(1+o(1)\big) \quad j=1,\dots,n \text{ as } z\to\infty \text{ in } \mathbf{C}\setminus E. \tag{2.1}$$

Set $H = \{|z| = r\colon\ z \in E,\ |z| > 1\}$. By Remark 2.1, $H$ is of finite logarithmic measure. By (1.7) and (2.1), we obtain that

$$\big[P_n(z) + \cdots + P_1(z) + P_0(z)\big] + \big[P_n(z) + \cdots + P_1(z)\big]o(1) = \frac{F(z)}{f(z)}. \tag{2.2}$$

By assumption, we see that

$$\deg(P_n + \cdots + P_1 + P_0) = k \geqslant \deg(P_n + \cdots + P_1) = s. \tag{2.3}$$

Set

$$P_n(z) + \cdots + P_1(z) + P_0(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0,$$
$$P_n(z) + \cdots + P_1(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0,$$

where $a_k, \dots, a_0, b_s, \dots, b_0$ are constants and $a_k b_s \neq 0$. We take $z$ satisfying $|z| = r \notin [0,1] \cup H$ and $|f(z)| = M(r, f)$. Since $f(z)$ is the transcendental entire function and $F(z)$ is the polynomial, we see that when $z$ satisfy $|z| = r \notin [0,1] \cup H$ and $|f(z)| = M(r, f)$,

$$\frac{F(z)}{f(z)} \to 0 \quad (z \to \infty). \tag{2.4}$$

If $s < k$, then by (2.4), we see (2.2) is a contradiction.
If $s = k$, then

$$\frac{P_n(z) + \cdots + P_1(z) + P_0(z)}{P_n(z) + \cdots + P_1(z)} \to \frac{a_k}{b_s} \neq 0 \quad (z \to \infty). \tag{2.5}$$

Thus, by (2.4) and (2.5), we see that (2.2) is also a contradiction. Hence $\sigma(f) \geqslant 1$ holds for Case 1.

**Case 2.** Suppose that $f(z)$ is a meromorphic function with infinitely many poles. Since $F(z), P_n(z), \dots, P_0(z)$ are polynomials, we see that there is a constant $M > 0$ such that all zeros of $F(z), P_n(z), \dots, P_0(z)$ are in $E_1 = \{z\colon\ |\operatorname{Re} z| < M,\ |\operatorname{Im} z| < M\}$.

Set

$$D_1 = \{z\colon\ \operatorname{Re} z > M_1\}; \qquad D_2 = \{z\colon\ \operatorname{Re} z < -M_1\};$$
$$D_3 = \{z\colon\ \operatorname{Im} z > M_1\}; \qquad D_4 = \{z\colon\ \operatorname{Im} z < -M_1\},$$

where $M_1 = M + n$. Since $f(z)$ has infinitely many poles, we see that there exists at least one of $D_j$ $(j = 1, \dots, 4)$, say $D_1$, such that $f(z)$ has infinitely many poles in $D_1$. Suppose that a point $z_0 \in D_1$ satisfies $f(z_0) = \infty$. Thus, there is $j_1 \in \{1, \dots, n\}$ satisfying $z_0 + j_1 \in D_1$ and $f(z_0 + j_1) = \infty$. Similarly, there is a sequence $\{j_d\colon\ d = 1, \dots\}$ satisfying $j_d \in \{1, \dots, n\}$ $(d = 1, \dots)$, $z_0 + j_1 + \cdots + j_d \in D_1$ and $z_0 + j_1 + \cdots + j_d$ are poles of $f(z)$. Since $|j_d| \leqslant n$ for $d = 1, \dots$ and $n$ is fixed, we see that $\lambda(\frac{1}{f}) \geqslant 1$, so that $\sigma(f) \geqslant 1$.

If $f(z)$ has infinitely many poles in $D_3$ (or $D_4$), then we may use the same method as above.

If $f(z)$ has infinitely many poles in $D_2$, then we can consider the other form of (1.7)

$$P_n(z-n) f(z) + P_{n-1}(z-n) f(z-1) + \cdots + P_0(z-n) f(z-n) = F(z-n),$$

and get a sequence $\{l_d\colon\ d = 1, \dots\}$ satisfying $l_d \in \{-1, \dots, -n\}$. So that $\lambda(\frac{1}{f}) \geqslant 1$, and $\sigma(f) \geqslant 1$.

Hence $\sigma(f) \geqslant 1$ holds for Case 2.

**Case 3.** Suppose that $f(z)$ is a transcendental function with finitely many poles. Thus, $f(z)$ can be rewritten as

$$f(z) = \frac{f_1(z)}{H(z)}$$

where $f_1(z)$ is a transcendental entire function, $H(z)$ is polynomial

$$H(z) = h z^m + h_{m-1} z^{m-1} + \cdots + h_0 \tag{2.6}$$

where $h \neq 0,\ h_{m-1}, \dots, h_0$ are constants. Substituting $f(z) = \frac{f_1(z)}{H(z)}$ into (1.7), we obtain that

$$P_n(z)\frac{f_1(z+n)}{H(z+n)} + P_{n-1}(z)\frac{f_1(z+n-1)}{H(z+n-1)} + \cdots + P_1(z)\frac{f_1(z+1)}{H(z+1)} + P_0(z)\frac{f_1(z)}{H(z)} = F(z). \tag{2.7}$$

Set

$$\begin{cases} P_j(z) = a_j z^{k_j} + a_{j,k_j-1} z^{k_j-1} + \cdots + a_{j,0}, \quad j = 0, \dots, n, \\ F(z) = c z^t + c_{t-1} z^{t-1} + \cdots + c_0, \end{cases} \tag{2.8}$$

where $a_j \neq 0$, $a_{j,k_j-1}, \dots, a_{j,0}$ $(j = 0, \dots, n)$ and $c \neq 0$, $c_{t-1}, \dots, c_0$ are constants, $\deg P_j = k_j \geqslant 0$ are integers. And set

$$\max\{\deg P_j\colon\ j = 0, \dots, n\} = k \geqslant 1. \tag{2.9}$$

Multiplying both sides of (2.7) by $H(z)\cdots H(z+n)$, we obtain that

$$Q_n(z)f_1(z+n)+Q_{n-1}(z)f_1(z+n-1)+\cdots+Q_1(z)f_1(z+1)+Q_0(z)f_1(z)=F_1(z), \tag{2.10}$$

where $Q_n,\dots,Q_0,\ F_1$ are polynomials:

$$\begin{cases} F_1(z)=F(z)H(z+n)\cdots H(z+1)H(z); \\ Q_n(z)=P_n(z)H(z+n-1)\cdots H(z+1)H(z); \\ Q_j(z)=P_j(z)H(z+n)\cdots H(z+j+1)H(z+j-1)\cdots H(z) \quad (j=1,\dots,n-1); \\ Q_0(z)=P_0(z)H(z+n)\cdots H(z+1). \end{cases} \tag{2.11}$$

By (2.6), (2.8) and (2.11), we obtain that

$$\begin{cases} F_1(z)=ch^{n+1}z^{t+(n+1)m}+\cdots; \\ Q_j(z)=a_jh^nz^{k_j+nm}+\cdots, \quad j=0,\dots,n. \end{cases} \tag{2.12}$$

By (2.9) and (2.12), we see that

$$\deg\bigl(Q_n(z)+\cdots+Q_0\bigr)=\deg\bigl(P_n(z)+\cdots+P_0(z)\bigr)+nm,$$

and

$$\begin{aligned} \max\{\deg Q_j\colon\ j=0,\dots,n\} &= \max\{\deg P_j\colon\ j=0,\dots,n\}+nm \\ &= k+nm=\deg\bigl(Q_n(z)+\cdots+Q_0\bigr). \end{aligned} \tag{2.13}$$

Since $f_1(z)$ is a transcendental entire function, by (2.13), we see that $f_1(z)$ satisfies Case 1. Hence

$$\sigma(f)=\sigma(f_1)\geqslant 1.$$

Thus, $\sigma(f)\geqslant 1$ holds for Case 3.

Finally, we prove that $\lambda(f)=\sigma(f)$. Set

$$E(z,f):=P_n(z)f(z+n)+\cdots+P_0(z)f(z)-F(z).$$

Thus,

$$E(z,0)=F(z)\not\equiv 0.$$

By Lemma 2.3, we have

$$m\left(r,\frac{1}{f}\right)=S(r,f),$$

so that

$$N\left(r,\frac{1}{f}\right)=T(r,f)+S(r,f).$$

Hence $\lambda(f)=\sigma(f)$. Theorem 1.1 is thus proved. □

## 3. The proof of Theorem 1.2

First, we suppose that $f(z)$ is a transcendental meromorphic solution of (1.8). We use a similar method as in the proof of Theorem 1.1 to prove Theorem 1.2. Suppose that $\sigma(f)<1$. By Lemma 2.2, there exists an $\varepsilon$-set $E$ such that

$$f(z+j)=f(z)\bigl(1+o(1)\bigr) \quad j=1,\dots,n \text{ as } z\to\infty \text{ in } \mathbf{C}\backslash E. \tag{3.1}$$

Set $H=\{|z|=r\colon\ z\in E,\ |z|>1\}$. By Remark 2.1, $H$ is of finite logarithmic measure. By (1.8) and (3.1), we obtain that

$$\bigl[P_n(z)+\cdots+P_1(z)+P_0(z)\bigr]+\bigl[P_n(z)+\cdots+P_1(z)\bigr]o(1)=0. \tag{3.2}$$

By assumption, we see that

$$\deg(P_n+\cdots+P_1+P_0)=k\geqslant\deg(P_n+\cdots+P_1)=s. \tag{3.3}$$

Using the same method as in the proof of Theorem 1.1, by (3.3), we see that (3.2) is a contradiction. Hence, if $f(z)$ is a transcendental meromorphic solution of (1.8), then $\sigma(f)\geqslant 1$.

Secondly, we prove that Eq. (1.8) has no non-zero rational solution. Since $\deg(P_n+\cdots+P_0)=\max\{\deg P_j\colon\ j=0,\dots,n\}\geqslant 1$, we clearly know that Eq. (1.8) has no non-zero constant solution. Now suppose that (1.8) has a non-constant rational solution

$$f(z)=\frac{h(z)}{H(z)}=\frac{cz^m+c_{m-1}z^{m-1}+\cdots+c_0}{dz^s+d_{s-1}z^{s-1}+\cdots+d_0}, \tag{3.4}$$

where $c \neq 0, c_{m-1}, \dots, c_0$, $d \neq 0, d_{s-1}, \dots, d_0$ are constants, $m, s$ are non-negative integers such that $m + s \geqslant 1$. So that, by (3.4), we have

$$f(z+j) = \frac{h(z+j)}{H(z+j)} = \frac{cz^m + c_{m-1}^{(j)} z^{m-1} + \dots + c_0^{(j)}}{dz^s + d_{s-1}^{(j)} z^{s-1} + \dots + d_0^{(j)}} \quad (j = 1, \dots, n), \tag{3.5}$$

where $c_{m-1}^{(j)}, \dots, c_0^{(j)}, d_{s-1}^{(j)}, \dots, d_0^{(j)}$ $(j = 1, \dots, n)$ are constants. Substituting (3.4) and (3.5) into (1.8), then eliminating denominators, we obtain that

$$P_n(z)A_n(z) + P_{n-1}(z)A_{n-1}(z) + \dots + P_1(z)A_1(z) + P_0(z)A_0(z) = 0, \tag{3.6}$$

where $A_n, \dots, A_0$ are polynomials:

$$\begin{cases} A_n(z) = h(z+n)H(z+n-1)\cdots H(z+1)H(z) = cd^n z^{m+ns} + \cdots; \\ A_j(z) = h(z+j)H(z+n)\cdots H(z+j+1)H(z+j-1)\cdots H(z) = cd^n z^{m+ns} + \cdots, \\ \quad j = 1, \dots, n-1; \\ A_0(z) = h(z)H(z+n)\cdots H(z+1) = cd^n z^{m+ns} + \cdots. \end{cases} \tag{3.7}$$

Since the first term of every $A_j$ is $cd^n z^{m+ns}$, by (3.6) and (3.7), we see that the left side of (3.6) is a polynomial with the degree

$$m + ns + \deg(P_0 + \dots + P_n) = m + ns + \max\{\deg P_j\colon\ j = 0, \dots, n\} \geqslant 2.$$

This is a contradiction. Hence (1.8) has no non-zero rational solution.

Finally, we prove that $f(z)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often and $\lambda(f-a) = \sigma(f)$. Set

$$E(z, f) := P_n(z)f(z+n) + \dots + P_0(z)f(z).$$

Suppose that $a \in \mathbf{C} \backslash \{0\}$. Thus, since $a \neq 0$ and (1.6), we have

$$E(z, a) = a\bigl(P_n(z) + \dots + P_0(z)\bigr) \not\equiv 0. \tag{3.8}$$

By Lemma 2.3 and (3.8), we have

$$m\left(r, \frac{1}{f-a}\right) = S(r, f),$$

so that

$$N\left(r, \frac{1}{f-a}\right) = T(r, f) + S(r, f).$$

Hence $\lambda(f-a) = \sigma(f)$. Theorem 1.2 is thus proved.

## 4. The proof of Theorem 1.3

Checking the proof for Case 2 in the proof of Theorem 1.1, we see that in the proof for Case 2, we do not apply the condition (1.6), so that, using the same method as in the proof for Case 2, we can prove Theorem 1.3.

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